Quick Introduction to the Fourier Transform

by Prof. Brian Evans
and Mr. Jean Farias
The University of Texas
at Austin Spring 2013

Fourier Transform / Hz / seconds

$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t}\, dt$$

real variable f

Amplitude real or complex

Real variable t

Amplitude is usually complex

real   imag

$$e^{-j2\pi f t} = \cos(-2\pi f t) + j\sin(-2\pi f t)$$

complex-valued sinusoid

$$= \cos(2\pi f t) - j\sin(2\pi f t)$$

$$x(t) \; \overset{\mathcal{F}}{\rightleftarrows} \; X(f)$$

$$x_1(t) + x_2(t) \; \overset{\mathcal{F}}{\rightleftarrows} \; X_1(f) + X_2(f)$$

$\cos(\theta)$ $\cos(-\theta) = \cos(\theta)$ EVEN

$\sin(-\theta) = -\sin\theta$ odd

$$\boxed{\bar{X}(s) = \mathcal{J}\{x(t)\}}$$
region of convergence (ROC)
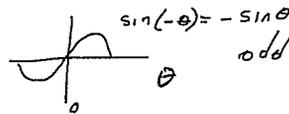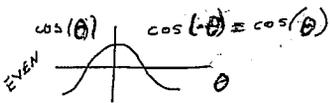
Laplace Transform

$$\bar{X}(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$$

If the ROC includes the imaginary axis

$$X(f) = \bar{X}(s)\Big|_{s = j2\pi f}$$

Example

Dirac delta functional — Continuous-Time Impulse

unit area

$(1)\; \delta(t)$

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1 \qquad \text{Unit Area}$$

Amplitude undefined at origin
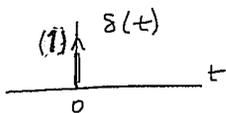
Mathematically models impulsive event at origin

$$\int_{-\infty}^{\infty} g(t)\, \delta(t)\, dt = g(0)$$

Unit area concentrated at origin — area denoted by parentheses

Sifting Property

if $g(0)$ is defined at origin

$$\mathcal{I}(f) = \int_{-\infty}^{\infty} \delta(t) \underbrace{e^{-j2\pi ft}}_{g(t)} \, dt = 1 \qquad \text{when } t = 0$$

$\mathcal{I}(f)$
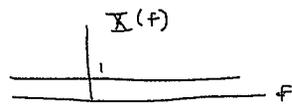
contains all
frequencies

How to get direct impulse response

$$\delta(t) \xrightarrow{} \boxed{\begin{array}{c} LTI \\ system \end{array}} \xrightarrow{h(t)}$$

impulse   Impulse
                 response
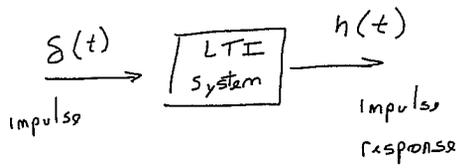
$$\delta(t) \xrightarrow{\mathcal{F}} 1$$

Example

two sided cosine

$\cos(2\pi f_0 t)$

this does not have laplace
it have fourier transform
con't build this in lab
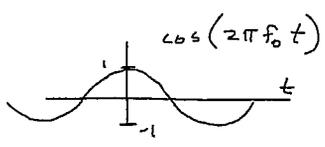
Euler's Formula
$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$\cos(2\pi f_0 t) = \frac{1}{2} e^{-j2\pi f_0 t} + \frac{1}{2} e^{j2\pi f_0 t}$$

$$= \frac{1}{2}\left(\cos(-2\pi f_0 t) + j\sin(-2\pi f_0 t)\right) + \frac{1}{2}\left(\cos(2\pi f_0 t) + j\sin(2\pi f_0 t)\right)$$

$$= \cos(2\pi f_0 t)$$

$$\int_{-\infty}^{\infty} e^{-j2\pi f_0 t} \, e^{-j2\pi ft} \, dt$$

$$= \int_{-\infty}^{\infty} e^{-j2\pi(f+f_0)t} \, dt = \delta(f+f_0)$$

Inverse Fourier Transform
$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} \, df$$

For $X(f) = \delta(f)$,
$$x(t) = \int_{-\infty}^{\infty} \delta(f) e^{j2\pi ft} \, df$$

$$= 1$$

$$1 \xrightarrow{\mathcal{F}} \delta(f)$$

$$X(f + f_o) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi(f + f_o)t}\, dt$$

$$= \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f_o t}\, e^{-j2\pi f t}\, dt$$

$$x(t)\, e^{-j2\pi f_o t} \quad \xrightarrow{\;\;\mathcal{F}\;\;} \quad X(f + f_o)$$

$$1 \quad \xrightarrow{\;\;\mathcal{F}\;\;} \quad \delta(f)$$

$$x(t) = \int_{-\infty}^{\infty} X(f)\, e^{j2\pi f t}\, df \qquad \text{Inverse Fourier}$$

$$X(f) = \delta(f), \qquad x(t) = \int_{-\infty}^{\infty} \underbrace{\delta(f)\, e^{j2\pi f t}}_{g(f)}\, df = g(o) = 1$$

always evaluate this at the origin in the above ⟩ case is when

$$f = 0$$

$$x(t)\, e^{-j2\pi f_o t} \quad \xrightarrow{\;\;\mathcal{F}\;\;} \quad X(f + f_o)$$

$$1 \quad \xrightarrow{\;\;\mathcal{F}\;\;} \quad \delta(f)$$

$$1 \cdot e^{-j2\pi f_o t} \quad \xrightarrow{\;\;\mathcal{F}\;\;} \quad \delta(f + f_o)$$

$$e^{j2\pi f_o t} \quad \xrightarrow{\;\;\mathcal{F}\;\;} \quad \delta(f - f_o)$$

$$\mathcal{F}\{\cos(2\pi f_o t)\} = \mathcal{F}\left\{ \tfrac{1}{2} e^{-j2\pi f_o t} + \tfrac{1}{2} e^{j2\pi f_o t} \right\}$$

$$= \tfrac{1}{2}\,\delta(f + f_o) + \tfrac{1}{2}\,\delta(f - f_o)$$

$$\underbrace{\hphantom{xxx}}_{\text{area of } \frac{1}{2}} \qquad \underbrace{\hphantom{xxx}}_{\text{area of } \frac{1}{2}}$$
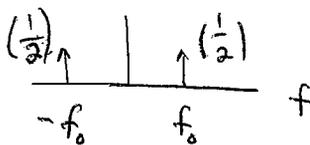
1ST find the origin

$$f + f_o = 0 \qquad f - f_o = 0$$

$$f = -f_o \qquad f = f_o$$

Example using the rectangular pulse

$rect(t)$
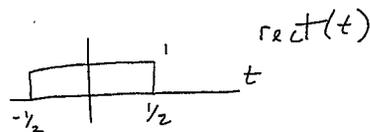
Has unit area.

Taking the Fourier transform,

$$\int_{-1/2}^{1/2} x(t)\, e^{-j2\pi f t}\, dt$$

$$\frac{1}{-j2\pi f}\, e^{-j2\pi f t}\,\Big|_{-1/2}^{1/2}$$

$$\frac{1}{-j2\pi f}\, e^{-j\pi f} + \frac{1}{j2\pi f}\, e^{j\pi f}$$

$$\frac{j}{j(-j)2\pi f}\, e^{-j\pi f} + \frac{j}{j(j)2\pi f}\, e^{j\pi f}$$

$$\frac{j}{2\pi f}\, e^{-j\pi f} - \frac{j}{2\pi f}\, e^{j\pi f}$$

$$\frac{j}{2\pi f}\left(e^{-j\pi f} - e^{j\pi f}\right)$$

$$\frac{j}{2\pi f}\left(-2j\sin(\pi f)\right) = \frac{\sin(\pi f)}{\pi f} = sinc(f)$$